Development Economics (PhD)
Intertemporal Utility Maximization

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- Consider a two-period (period 1 and period 2) problem
- The two-period utility function:

$$U = u(c_1) + \frac{1}{1+\delta}u(c_2) \tag{1}$$

- Standard assumption: $u'(.) > 0$, and $u''(.) < 0$
- Period 1 budget constraint:

$$Y_1 + (1+r)A_0 = c_1 + A_1 \tag{2}$$

- $A_0 > 0, \implies$ inheritance
- $A_0 < 0, \implies$ debt from previous generation
- Period 2 budget constraint:

- Period 2 budget constraint:

$$Y_2 = (1+r)A_1 = c_2 + A_2 \tag{3}$$

- For simplicity, assume $A_0 = 0, \Longrightarrow$ no inheritance, and no debt; $A_2 = 0$ no bequest/no debt in the final year $(A_2) \Longrightarrow$

$$max_{c_1,c_2} \;\; U = u(c_1) + \frac{1}{1+\delta}u(c_2) \qquad s.t \tag{4}$$

$$Y_1 = c_1 + A_1 \tag{5}$$

$$Y_2 + (1+r)A_1 = c_2 \tag{6}$$

Two Period Utility Maximization
Lagrange Multiplier Method Cont.

- Use the Lagrange multiplier method

$$L = u(c_1) + \frac{1}{1+\delta}u(c_2) + \lambda_1(Y_1 - c_1 - A_1)$$

$$+\lambda_2(Y_2 + (1+r)A_1 - c_2) \tag{7}$$

- FOCs for c_1, c_2, A_1, λ_1 and λ_2

$$\frac{\partial L}{\partial c_1} = u'(c_1) - \lambda_1 = 0 \tag{8}$$

$$\frac{\partial L}{\partial c_2} = \frac{1}{1+\delta}u'(c_2) - \lambda_2 = 0 \tag{9}$$

Two Period Utility Maximization
Lagrange Multiplier Method Cont.

$$\frac{\partial L}{\partial A_1} = -\lambda_1 + (1+r)\lambda_2 = 0 \tag{10}$$

$$\frac{\partial L}{\partial \lambda_1} = Y_1 = c_1 + A_1 \tag{11}$$

$$\frac{\partial L}{\partial \lambda_2} = Y_2 + (1+r)A_1 = c_2 \tag{12}$$

- $(8) - (10) \Longrightarrow$

$$\lambda_1 = u'(c_1) \tag{13}$$

$$\lambda_2 = \frac{1}{1+\delta}u'(c_2) \quad and \tag{14}$$

Two Period Utility Maximization
Lagrange Multiplier Method Cont.

$$-\lambda_1 + (1+r)\lambda_2 = 0 \tag{15}$$

- Substituting equations (13) and (14) into (15) gives

$$u'(c_1) = \frac{1+r}{1+\delta} u'(c_2) \tag{16}$$

- Eq. (16) is known as the **Euler Equation**

- Consider the lifetime utility function of the consumer, $U =:$

$$u(c_1) + \frac{1}{1+\delta}u(c_2) + (\frac{1}{1+\delta})^2 u(c_3) + ... + (\frac{1}{1+\delta})^{T-1} u(c_T) \tag{17}$$

$$[= \sum_{t=1}^{T} (\frac{1}{1+\delta})^{T-1} u(c_t)] \quad s.t \tag{18}$$

- The optimization problem of the consumer is given as follows:

Finite Horizon Utility Maximization

Dynamic Programming Cont.

$$max_{c_t} \; [U = \sum_{t=1}^{T} (\frac{1}{1+\delta})^{T-1} u(c_t)] \quad s.t \tag{19}$$

$$A_t = (1+r)A_{t-1} + Y_t - c_t \tag{20}$$

- In dynamic optimization, A_t is called the **State Variable**, representing the total amount of resources available to the consumer
- c_t is called **control variable** which must be chosen by the consumer to maximize utility
- Note that the level of consumption chosen at any time t (given A_{t-1}) affects the level of wealth available in period $t+1$
- The optimization problem of the consumer would be solved recursively (from period 1 to T)

- Let the value function (the maximized value of the objective function) in $t = 1$ given an initial stock of wealth A_0 be given by

$$V_1(A_0) = max_{c_1} \sum_{t=1}^{T} (\frac{1}{1+\delta})^{T-1} u(c_t) \tag{21}$$

$$V_1(A_0) = max_{c_1} \{u(c_1) + \frac{1}{1+\delta} u(c_2) + (\frac{1}{1+\delta})^2 u(c_3) + ...$$

$$+(\frac{1}{1+\delta})^{T-1} u(c_T)\} \tag{22}$$

$$V_1(A_0) = max_{c_1} \; \{u(c_1) + \frac{1}{1+\delta}[\sum_{t=2}^{T}(\frac{1}{1+\delta})^{T-2}u(c_t)]\} \quad s.t \tag{23}$$

$$A_t = (1+r)A_{t-1} + Y_t - c_t \tag{24}$$

- Given an initial stock of wealth from period 1, A_1 the consumer again maximizes utility in period 2 subject to the wealth constraint given in (24) above
- Thus the value function in period 2 would be

$$V_2(A_1) = max_{c_2} \; \sum_{t=2}^{T}(\frac{1}{1+\delta})^{T-2}u(c_t) \tag{25}$$

- Substituting equation (25) into (23) yields

Finite Horizon Utility Maximization
Dynamic Programming Cont.

$$V_1(A_0) = max_{c_1}\ \left[u(c_1) + \frac{1}{1+\delta} V_2(A_1)\right] \tag{26}$$

- If the consumer optimizes in such a way,

$$V_t(A_{t-1}) = max_{c_t}\ \left[u(c_t) + \frac{1}{1+\delta} V_{t+1}(A_t)\right] \tag{27}$$

- Equation (27) is known as the **Bellman Equation**
- Drop the subscript t in the value function

$$V(A_{t-1}) = max_{c_t}\ \left[u(c_t) + \frac{1}{1+\delta} V(A_t)\right] \tag{28}$$

- Differentiate the Bellman equation w.r.t c_t

$$\frac{\partial V(A_{t-1})}{\partial c_t} = 0 \tag{29}$$

$$\iff u'(c_t) + \frac{1}{1+\delta} V'(A_t) \frac{\partial A_t}{\partial c_t} = 0 \tag{30}$$

- Since $\partial A_t / \partial c_t = -1$ from e.q (24), we can rewrite (30) as

$$u'(c_t) - \frac{1}{1+\delta} V'(A_t) = 0 \tag{31}$$

- We need to solve for $V'(A_t)$ in eq (31) to complete the optimization
- Differentiate the Bellman equation given in (28) w.r.t A_{t-1}

Finite Horizon Utility Maximization

Dynamic Programming Cont.

$$V'(A_{t-1}) = (\frac{1}{1+\delta})V'(A_t)\frac{\partial A_t}{\partial A_{t-1}} \tag{32}$$

$$V'(A_{t-1}) = (\frac{1+r}{1+\delta})V'(A_t) \tag{33}$$

- Eq (31) $\Longrightarrow$:

$$V'(A_t) = (1+\delta)u'(c_t) \tag{34}$$

- Eq. (34) $\Longrightarrow$

$$V'(A_{t-1}) = (1+\delta)u'(c_{t-1}) \tag{35}$$

- Substituting (34) and (35) into (33) gives

$$u'(c_t) = \frac{1+r}{1+\delta}u'(c_{t+1}) \tag{36}$$

Finite Horizon Utility Maximization
Dynamic Programming Cont.

- If $\delta = r$,

$$u'(c_t) = u'(c_{t+1}) \tag{37}$$